The dynamical additivity and the strong dynamical additivity of quantum operations

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Abstract

In the paper, the dynamical additivity of bi-stochastic quantum operations is characterized and the strong dynamical additivity is obtained under some restrictions.

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1 Introduction

In quantum information theory, there are two well-known entropic inequalities for quantum-mechanical systems, that is, subadditivity of entropy of bipartite quantum state ρ^{AB}:

$$\mathsf{S}(\rho^{AB}) \leqslant \mathsf{S}(\rho^{A}) + \mathsf{S}(\rho^{B})$$

and strong subadditivity of entropy of tripartite quantum state ABC:

$$\mathsf{S}(\rho^{ABC}) + \mathsf{S}(\rho^{B}) \leqslant \mathsf{S}(\rho^{AB}) + \mathsf{S}(\rho^{BC}),$$

where $\rho^X(X = A, B, AB, BC)$ are the reduction to corresponding system X. In quantum information processing, one are especially interested in the extreme cases of quantum states, for instance, under what conditions the subadditivity or strong subadditivity inequality of entropy of quantum states are saturated? By the Pinsker's inequality [13]:

$$\mathsf{S}(\rho^{A}) + \mathsf{S}(\rho^{B}) - \mathsf{S}(\rho^{AB}) \geqslant \frac{1}{2\ln 2}\left(\left\|\rho^{AB} - \rho^{A}\otimes\rho^{B}\right\|_1\right)^2,$$

where $\|*\|_1$ are the trace-norm, it follows that $\mathsf{S}(\rho^{AB}) = \mathsf{S}(\rho^{A}) + \mathsf{S}(\rho^{B})$ if and only if $\rho^{AB} = \rho^{A}\otimes\rho^{B}$. This resolves the saturation of subadditivity inequality. Compared with the subadditivity, the more complicated construction that follows will give the solution to the saturation of the strong subadditivity inequality. The description is as follows [8]: a tripartite state ρ^{ABC} are such that $\mathsf{S}(\rho^{AB}) + \mathsf{S}(\rho^{BC}) = \mathsf{S}(\rho^{ABC}) + \mathsf{S}(\rho^{B})$ if and only if there is a decomposition of Hilbert space $\mathcal{H}^B$ which is used to describe the system B:

$$\mathcal{H}^B = \bigoplus_k \mathcal{H}_k^L \otimes \mathcal{H}_k^R$$

into a direct sum of tensor products such that

$$\rho^{ABC} = \bigoplus_k p_k \rho_k^{AL} \otimes \rho_k^{RC},$$

where ρ_k^{AL} is a state on $\mathcal{H}^A \otimes \mathcal{H}_k^L$ and ρ_k^{RC} is a state on $\mathcal{H}_k^R \otimes \mathcal{H}^C$ for each index k and $\{p_k\}$ is a probability distribution.

Similarly these problems above-mentioned can be considered in the regime of quantum operations. Let Φ, Λ and Ψ be three stochastic quantum operations (the notations will be explained later) on a quantum system space $\mathcal{H}$. The study on the behavior of map entropy of composition of stochastic quantum operations is an important and interesting problem. Recently Roga *et. al.* [15] obtained that if Φ is bi-stochastic, then we have the *dynamical subadditivity*:

$$\mathsf{S}(\Phi \circ \Psi) \leqslant \mathsf{S}(\Phi) + \mathsf{S}(\Psi).$$

Moreover, if Φ, Λ and Ψ are all bi-stochastic, then we have the *strong dynamical subadditivity*:

$$\mathsf{S}(\Phi \circ \Lambda \circ \Psi) + \mathsf{S}(\Lambda) \leqslant \mathsf{S}(\Phi \circ \Lambda) + \mathsf{S}(\Lambda \circ \Psi).$$

In this paper, motivated by the structure of states which saturate the inequality of strong subadditivity of quantum entropy, we discuss under what conditions the dynamical subadditivity and the strong dynamical subadditivity can be saturated, that is, the dynamical additivity and the strong dynamical additivity. Firstly, by using entropy-preserving extensions of quantum states, a characterization of dynamical additivity of bi-stochastic quantum operations is obtained. Next, we show that if quantum operations are local operations [4, 6] and have some kind of orthogonality, then the strong dynamical additivity is also true.

2 Preliminaries

In this section we clarify the notations used in our paper. Throughout the paper, only finite-dimensional Hilbert spaces $\mathcal{H}$ are considered. Let $\mathbf{L}(\mathcal{H})$ be the set of all linear operators from $\mathcal{H}$ to $\mathcal{H}$. A state ρ of some quantum system, described by $\mathcal{H}$, is a positive semi-definite operator of trace one, in particular, for each unit vector $|\psi\rangle \in \mathcal{H}$, the operator $\rho = |\psi\rangle\langle\psi|$ is said to be a *pure state*. The set of all states on $\mathcal{H}$ is denoted by $\mathsf{D}(\mathcal{H})$.

If $X, Y \in \mathbf{L}(\mathcal{H})$, then $\langle X, Y\rangle = \mathrm{Tr}(X^\dagger Y)$ defines an inner product on $\mathbf{L}(\mathcal{H})$, which is called the *Hilbert-Schmidt inner product*. It is easily seen that if $X, Y \in \mathbf{L}(\mathcal{H})$ are two positive semi-definite operators, X and Y are orthogonal, i.e., $\langle X, Y\rangle = 0$, if and only if $XY = 0$.

Let $S, T \in \mathbf{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ be two positive semi-definite operators, where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Denote $Y_1 = \mathrm{Tr}_2(Y), Y_2 = \mathrm{Tr}_1(Y)(Y = S, T)$. Then $S_1, T_1, S_2, T_2 \in \mathbf{L}(\mathcal{H})$ are all positive semi-definite operators. If $S_1T_1 = S_2T_2 = 0$, then S and T are said to be *bi-orthogonal* [9]. Thus the notion of a state decomposition that is bi-orthogonal is defined as follows:

Definition 2.1. ([9]) Let ρ^{AB} be a bipartite state in $\mathsf{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The following state decomposition $\rho^{AB} = \sum_k p_k\rho_k^{AB}$, where $\rho_k^{AB} \in \mathsf{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ for each k and $\{p_k\}$ is a probability distribution with each $p_k > 0$, is called *bi-orthogonal* if, in terms of the reductions of ρ_k^{AB}, $\rho_k^X\rho_{k'}^X = 0(X = A, B; \forall k \neq k')$.

Let $\{|m\rangle\}$ be the standard basis for $\mathcal{H}_2$, correspondingly $\{|\mu\rangle\}$ for $\mathcal{H}_1$. For each $P = \sum_{m,\mu} p_{m\mu}|m\rangle\langle\mu| \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$, if we denote $\mathrm{vec}(P) = \sum_{m,\mu} p_{m\mu}|m\mu\rangle$, then **vec** defines a simple correspondence between $\mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{H}_2 \otimes \mathcal{H}_1$. Moreover, if $\mathcal{H}_A$ and $\mathcal{H}_B$ are two Hilbert spaces, $\{|m\rangle\}$ and $\{|\mu\rangle\}$ are their standard bases, respectively, then we can also define a map **vec** over a bipartite space that describes a change of bases from the standard basis of $\mathbf{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ to the standard basis of $\mathcal{H}_A \otimes \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_B$, that is,

$$\mathrm{vec}(|m\rangle\langle n| \otimes |\mu\rangle\langle\nu|) = |mn\rangle \otimes |\mu\nu\rangle.$$

The following properties of the **vec** map are easily verified [17]:

1. The **vec** map is a linear bijection. It is also an isometry, in the sense that

$$\langle X, Y\rangle = \langle \mathrm{vec}(X), \mathrm{vec}(Y)\rangle$$

for all $X, Y \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$.

2. For every choice of operators $A \in \mathbf{L}(\mathcal{H}_1, \mathcal{K}_1)$, $B \in \mathbf{L}(\mathcal{H}_2, \mathcal{K}_2)$, and $X \in \mathbf{L}(\mathcal{H}_2, \mathcal{H}_1)$, it holds that

$$(A \otimes B)\,\mathrm{vec}(X) = \mathrm{vec}(AXB^{\mathsf{T}}).$$

3. For every choice of operators $A, B \in \mathbf{L}(\mathcal{H}_1, \mathcal{H}_2)$, the following equations hold:

$$\begin{aligned} \mathrm{Tr}_1(\mathrm{vec}(A)\,\mathrm{vec}(B)^{\dagger}) &= AB^{\dagger}, \\ \mathrm{Tr}_2(\mathrm{vec}(A)\,\mathrm{vec}(B)^{\dagger}) &= (B^{\dagger}A)^{\mathsf{T}}. \end{aligned}$$

4. If $X \in \mathbf{L}(\mathcal{H}_A)$, $Z \in \mathbf{L}(\mathcal{H}_B)$, then $\mathrm{vec}(X \otimes Z) = \mathrm{vec}(X) \otimes \mathrm{vec}(Z)$.

Denote by $\mathsf{T}(\mathcal{H})$ the set of all *linear super-operators* from $\mathbf{L}(\mathcal{H})$ to $\mathbf{L}(\mathcal{H})$. For each $\Phi \in \mathsf{T}(\mathcal{H})$, it follows from the Hilbert-Schmidt inner product of $\mathbf{L}(\mathcal{H})$ that there is a linear super-operator $\Phi^{\dagger} \in \mathsf{T}(\mathcal{H})$ such that $\langle \Phi(X), Y\rangle = \langle X, \Phi^{\dagger}(Y)\rangle$ for any $X, Y \in \mathbf{L}(\mathcal{H})$. $\Phi^{\dagger}$ is referred to the *dual super-operator* of Φ.

$\Phi \in \mathsf{T}(\mathcal{H})$ is said to be *completely positive* (CP) if for each $k \in \mathbb{N}$, $\Phi \otimes \mathbb{1}_{M_k(\mathbb{C})} : \mathbf{L}(\mathcal{H}) \otimes M_k(\mathbb{C}) \to \mathbf{L}(\mathcal{H}) \otimes M_k(\mathbb{C})$ is positive, where $M_k(\mathbb{C})$ is the set of all $k \times k$ complex matrices. It follows from the famous theorems of Choi [3] and Kraus [11] that Φ can be represented in the following form: $\Phi = \sum_j Ad_{M_j}$, where $\{M_j\}_{j=1}^n \subseteq \mathbf{L}(\mathcal{H})$, that is, $\Phi(X) = \sum_{j=1}^n M_j X M_j^{\dagger}$, $X \in \mathbf{L}(\mathcal{H})$. Throughout this paper, $\dagger$ means the adjoint operation of an operator. Moreover, if $\{M_j\}_{j=1}^n$ is pairwise orthogonal, then $\Phi = \sum_j Ad_{M_j}$ is said to be a canonical representation of Φ. In [3, 10], it was proved that each quantum operation has a canonical representation.

The so-called *quantum operation* on $\mathcal{H}$ is just a CP and trace non-increasing super-operator $\Phi \in \mathsf{T}(\mathcal{H})$, moreover, if Φ is CP and trace-preserving, then it is called *stochastic*; if Φ is stochastic and unit-preserving, then it is called *bi-stochastic*.

The famous *Jamiołkowski isomorphism* $J : \mathsf{T}(\mathcal{H}) \longrightarrow \mathbf{L}(\mathcal{H} \otimes \mathcal{H})$ transforms each $\Phi \in \mathsf{T}(\mathcal{H})$ into an operator $J(\Phi) \in \mathbf{L}(\mathcal{H} \otimes \mathcal{H})$, where $J(\Phi) = \Phi \otimes \mathbb{1}_{\mathbf{L}(\mathcal{H})}(\mathrm{vec}(\mathbb{1}_{\mathcal{H}})\,\mathrm{vec}(\mathbb{1}_{\mathcal{H}})^{\dagger})$. If $\Phi \in \mathsf{T}(\mathcal{H})$ is CP, then $J(\Phi)$ is a positive semi-definite operator, in particular, if Φ is stochastic, then $\frac{1}{N}J(\Phi)$ is a state on $\mathcal{H} \otimes \mathcal{H}$, we denote the state by $\rho(\Phi)$, [1].

The information encoded in a quantum state $\rho \in \mathsf{D}(\mathcal{H})$ is quantified by its *von Neumann entropy* $\mathsf{S}(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$. If $\Phi \in \mathsf{T}(\mathcal{H})$ is a stochastic quantum operation, we denote the von Neumann entropy $\mathsf{S}(\rho(\Phi))$ of $\rho(\Phi)$ by $\mathsf{S}(\Phi)$ which is called *map entropy*, $\mathsf{S}(\Phi)$ describes the decoherence induced by the quantum operation Φ.

3 Entropy-Preserving Extensions of Quantum States and the Dynamical Additivity

The technique of quantum state extension without changing entropy is a very important and useful tool. It is employed by Datta to construct an example which shows equivalence of the positivity of quantum discord and strong subadditivity for quantum mechanical systems. Based on this fact, Datta obtained that zero discord states are precisely those states which satisfy the strong additivity for quantum mechanical systems [7]. In what follows, we will use it to give a characterization of dynamical additivity of map entropy.

The next proposition is concerned with one type of quantum state extensions without changing entropy.

Proposition 3.1. *Let $\rho \in D(\mathcal{H})$. If $\{|i\rangle\}$ is a basis for $\mathcal{H}$ and $\rho = \sum_{i,j=1}^{N} \rho_{i,j}|i\rangle\langle j|$, then $\widetilde{\rho} = \sum_{i,j=1}^{N} \rho_{i,j}|ii\rangle\langle jj|$ is a state in $D(\mathcal{H}\otimes\mathcal{H})$, and $S(\widetilde{\rho}) = S(\rho)$.*

Proof. By the spectral decomposition theorem, $\rho = \sum_k \lambda_k |x_k\rangle\langle x_k|$, where $\lambda_k \geqslant 0$, $\{|x_k\rangle\}$ is an orthonormal set for $\mathcal{H}$. This implies that $\rho_{i,j} = \langle i|\rho|j\rangle = \sum_k \lambda_k \langle i|x_k\rangle\langle x_k|j\rangle = \sum_k \lambda_k x_k^{(i)} \overline{x}_k^{(j)}$. Note that $\{|x_k\rangle\}$ is an orthonormal set for $\mathcal{H}$, so $\sum_{i=1}^{N} x_m^{(i)} \overline{x}_n^{(i)} = \delta_{mn}$. Now

$$
\begin{aligned}
\widetilde{\rho} &= \sum_{i,j=1}^{N} \Big(\sum_k \lambda_k x_k^{(i)} \overline{x}_k^{(j)}\Big)|i\rangle\langle j| \otimes |i\rangle\langle j| = \sum_k \lambda_k \Big(\sum_{i,j=1}^{N} x_k^{(i)} \overline{x}_k^{(j)} |i\rangle\langle j| \otimes |i\rangle\langle j|\Big) \\
&= \sum_k \lambda_k \Big(\sum_{i=1}^{N} x_k^{(i)}|ii\rangle\Big)\Big(\sum_{i=1}^{N} x_k^{(i)}|ii\rangle\Big)^{\dagger} = \sum_k \lambda_k \operatorname{vec}(X_k)\operatorname{vec}(X_k)^{\dagger},
\end{aligned}
$$

where $\operatorname{vec}(X_k) = \sum_{i=1}^{N} x_k^{(i)}|ii\rangle \in \mathcal{H}\otimes\mathcal{H}$. Moreover, it is easy to show that $\operatorname{vec}(X_m)^{\dagger}\operatorname{vec}(X_n) = \delta_{mn}$, thus $\widetilde{\rho}$ is a state on $\mathcal{H}\otimes\mathcal{H}$. It is obvious that $\mathsf{S}(\widetilde{\rho}) = \mathsf{S}(\rho)$. ◻

Let $\Lambda \in \mathsf{T}(\mathcal{H})$ be stochastic. If Λ has two Kraus representations $\Lambda = \sum_{p=1}^{d_1} Ad_{S_p} = \sum_{q=1}^{d_2} Ad_{T_q}$, $\rho \in \mathsf{D}(\mathcal{H})$, take two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that $\dim \mathcal{H}_1 = d_1$, $\dim \mathcal{H}_2 = d_2$, $\{|m\rangle\}$ and $\{|\mu\rangle\}$ are the base of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Define

$$
\gamma_1(\Lambda) = \sum_{m,n=1}^{d_1} \operatorname{Tr}(S_m \rho S_n^{\dagger})|m\rangle\langle n| \text{ and } \gamma_2(\Lambda) = \sum_{\mu,\nu=1}^{d_2} \operatorname{Tr}(T_\mu \rho T_\nu^{\dagger})|\mu\rangle\langle \nu|.
$$

Then $\gamma_k \in \mathsf{D}(\mathcal{H}_k)(k = 1,2)$, and $\mathsf{S}(\gamma_1(\Lambda)) = \mathsf{S}(\gamma_2(\Lambda))$.

In fact, without loss of generality, we may assume $d_1 = d_2 = d$. Then there exists a $d\times d$ unitary matrix $U = [u_{m\mu}]$ such that for each $1 \le m \le d$, $S_m = \sum_{\mu=1}^{d} u_{m\mu} T_\mu$. Thus

$$
\begin{aligned}
\sum_{m,n=1}^{d} \operatorname{Tr}(S_m \rho S_n^{\dagger})|m\rangle\langle n| &= \sum_{m,n=1}^{d} \operatorname{Tr}\Big(\Big(\sum_{\mu=1}^{d} u_{m\mu}T_\mu\Big)\rho\Big(\sum_{\mu=1}^{d} u_{n\mu}T_\mu\Big)^{\dagger}\Big)|m\rangle\langle n| \\
&= U\left[\sum_{\mu,\nu=1}^{d} \operatorname{Tr}(T_\mu \rho T_\nu^{\dagger})|m\rangle\langle n|\right]U^{\dagger}.
\end{aligned}
$$

Let $V : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a unitary operator such that $V|m\rangle = |\mu\rangle$. Then

$$
\sum_{m,n=1}^{d} \operatorname{Tr}(S_m \rho S_n^{\dagger})|m\rangle\langle n| = UV\left[\sum_{\mu,\nu=1}^{d} \operatorname{Tr}(T_\mu \rho T_\nu^{\dagger})|\mu\rangle\langle \nu|\right]V^{\dagger}U^{\dagger},
$$

which implies that γ_1 and γ_2 are unitarily equivalent and thus the conclusion follows [14].

For each stochastic $\Lambda \in \mathsf{T}(\mathcal{H})$ and $\rho \in \mathsf{D}(\mathcal{H})$, we denote $\mathsf{S}(\rho;\Lambda)$ by $\mathsf{S}(\gamma_1(\Lambda))$. It follows from the above discussion that $\mathsf{S}(\rho;\Lambda)$ is well-defined [12]. Moreover, it is easy to see that if $\rho = \frac{1}{N}\mathbb{1}$, then $\mathsf{S}(\rho;\Lambda) = \mathsf{S}(\Lambda)$, [15].

It follows from above that if $\Phi, \Psi \in \mathsf{T}(\mathcal{H})$ are two bi-stochastic quantum operations, $\Phi = \sum_{m=1}^{N^2} Ad_{S_m}$ and $\Psi = \sum_{\mu=1}^{N^2} Ad_{T_\mu}$ are their canonical representations, respectively. Taking a N^2 dimensional complex Hilbert space $\mathcal{H}_0$, for each $\rho \in \mathsf{D}(\mathcal{H})$, we define

$$
\gamma(\Phi\circ\Psi) = \sum_{m,n,\mu,\nu=1}^{N^2} \operatorname{Tr}(S_m T_\mu \rho (S_n T_\nu)^{\dagger})|m\mu\rangle\langle n\nu|,
$$

then $\gamma(\Phi\circ\Psi)$ is a state on $\mathcal{H}_0\otimes\mathcal{H}_0$, and when $\rho = \frac{1}{N}\mathbb{1}$, $\mathsf{S}(\gamma(\Phi\circ\Psi)) = \mathsf{S}(\Phi\circ\Psi)$, that is, $\mathsf{S}(\rho, \Phi\circ\Psi) = \mathsf{S}(\Phi\circ\Psi)$.

Our mail result of this section is the following:

Theorem 3.2. *Let $\Phi, \Psi \in \mathsf{T}(\mathcal{H})$ be two bi-stochastic quantum operations, $\Phi(\rho) = \sum_{m=1}^{N^2} Ad_{S_m}$ and $\Psi = \sum_{\mu=1}^{N^2} Ad_{T_\mu}$ be their canonical representations, respectively. Then $\mathsf{S}(\Phi \circ \Psi) = \mathsf{S}(\Phi) + \mathsf{S}(\Psi)$ if and only if $\mathrm{Tr}(S_m T_\mu (S_n T_\nu)^\dagger) = \frac{1}{N} \mathrm{Tr}(S_m S_n^\dagger) \mathrm{Tr}(T_\mu T_\nu^\dagger)$; i.e., $\langle S_n T_\nu, S_m T_\mu \rangle = \frac{1}{N} \langle S_n, S_m \rangle \langle T_\nu, T_\mu \rangle$ for all $m, n, \mu, \nu = 1, \ldots, N^2$.*

Proof. The Jamiołkowski isomorphisms of Φ and Ψ are

$$J(\Phi) = \sum_{m=1}^{N^2} \mathrm{vec}(S_m)\, \mathrm{vec}(S_m)^\dagger, \; J(\Psi) = \sum_{\mu=1}^{N^2} \mathrm{vec}(T_\mu)\, \mathrm{vec}(T_\mu)^\dagger,$$

respectively, where $\langle \mathrm{vec}(S_m), \mathrm{vec}(S_n) \rangle = s_m \delta_{mn}$ and $\langle \mathrm{vec}(T_\mu), \mathrm{vec}(T_\nu) \rangle = t_\mu \delta_{\mu\nu}$. For each $\rho \in \mathsf{D}(\mathcal{H})$, let

$$\gamma(\Phi \circ \Psi) = \sum_{m,n,\mu,\nu=1}^{N^2} \mathrm{Tr}(S_m T_\mu \rho (S_n T_\nu)^\dagger) |m\mu\rangle\langle n\nu| = \sum_{m,n,\mu,\nu=1}^{N^2} \mathrm{Tr}(S_m T_\mu \rho (S_n T_\nu)^\dagger) |m\rangle\langle n| \otimes |\mu\rangle\langle \nu|.$$

Then we have

$$\begin{aligned} \gamma(\Psi) &= \sum_{\mu,\nu=1}^{N^2} \mathrm{Tr}(T_\mu \rho T_\nu^\dagger) |\mu\rangle\langle\nu| = \mathrm{Tr}_1(\gamma(\Phi \circ \Psi)), \\ \gamma(\Phi) &= \sum_{m,n=1}^{N^2} \mathrm{Tr}(S_m \rho S_n^\dagger)) |m\rangle\langle n| = \mathrm{Tr}_2(\gamma(\Phi \circ \Psi)). \end{aligned}$$

Note that when $\rho = \frac{1}{N}\mathbb{1}$, $\mathsf{S}(\gamma(\Phi \circ \Psi)) = \mathsf{S}(\Phi \circ \Psi)$, $\mathsf{S}(\gamma(\Psi)) = \mathsf{S}(\Psi)$ and $\mathsf{S}(\gamma(\Phi)) = \mathsf{S}(\Phi)$. Thus, we have

$$\begin{aligned} \mathsf{S}(\Phi \circ \Psi) = \mathsf{S}(\Phi) + \mathsf{S}(\Psi) \quad &\Leftrightarrow \quad \mathsf{S}(\gamma(\Phi)) + \mathsf{S}(\gamma(\Psi)) = \mathsf{S}(\gamma(\Phi \circ \Psi)) \\ &\Leftrightarrow \quad \gamma(\Phi \circ \Psi) = \gamma(\Phi) \otimes \gamma(\Psi) \\ &\Leftrightarrow \quad \mathrm{Tr}(S_m T_\mu (S_n T_\nu)^\dagger) = \frac{1}{N} \mathrm{Tr}(S_m S_n^\dagger) \mathrm{Tr}(T_\mu T_\nu^\dagger) \\ &\qquad = \frac{s_m t_\mu}{N} \delta_{mn} \delta_{\mu\nu} (\forall m, n, \mu, \nu = 1, \ldots, N^2). \end{aligned}$$

□

4 Bi-orthogonal Decomposition and Strong Dynamical Additivity

In order to study the strong dynamical additivity, we need the following bi-orthogonality and the bi-orthogonal decomposition of quantum operations.

Let $\Phi, \Psi \in \mathsf{T}(\mathcal{H})$ be CP super-operators. If their Jamiołkowski isomorphisms $J(\Phi)$ and $J(\Psi)$ are bi-orthogonal, then Φ and Ψ are said to be *bi-orthogonal*.

Proposition 4.1. *If $\Phi = \sum_\mu Ad_{M_\mu}$, $\Psi = \sum_\nu Ad_{N_\nu}$, then Φ and Ψ are bi-orthogonal if and only if $M_\mu^\dagger N_\nu = 0$ and $M_\mu N_\nu^\dagger = 0$ for all μ and ν, if and only if $\Phi \circ \Psi^\dagger = 0$ and $\Phi^\dagger \circ \Psi = 0$, if and only if $\Psi \circ \Phi^\dagger = 0$ and $\Psi^\dagger \circ \Phi = 0$.*

Proof. Note that $J(\Phi) = \sum_\mu \mathrm{vec}(M_\mu)\, \mathrm{vec}(M_\mu)^\dagger$, $J(\Psi) = \sum_\nu \mathrm{vec}(N_\nu)\, \mathrm{vec}(N_\nu)^\dagger$. By the definition, Φ and Ψ are bi-orthogonal if and only if $J(\Phi)$ and $J(\Psi)$ are bi-orthogonal, i.e., $\mathrm{Tr}_2(J(\Phi))\, \mathrm{Tr}_2(J(\Psi)) = \mathrm{Tr}_1(J(\Phi))\, \mathrm{Tr}_1(J(\Psi)) = 0$. Since

$$\mathrm{Tr}_2(J(\Phi))\, \mathrm{Tr}_2(J(\Psi)) \quad = \quad \left\{ \sum_\mu M_\mu {M_\mu}^\dagger \right\} \left\{ \sum_\nu N_\nu {N_\nu}^\dagger \right\} = \sum_{\mu,\nu} M_\mu {M_\mu}^\dagger N_\nu {N_\nu}^\dagger$$

and

$$\mathrm{Tr}_1(J(\Phi))\,\mathrm{Tr}_1(J(\Psi)) \quad = \quad \left\{\sum_\mu [M_\mu{}^\dagger M_\mu]^\mathsf{T}\right\}\left\{\sum_\nu [N_\nu{}^\dagger N_\nu]^\mathsf{T}\right\} = \sum_{\mu,\nu}[M_\mu{}^\dagger M_\mu]^\mathsf{T}[N_\nu{}^\dagger N_\nu]^\mathsf{T},$$

it follows that both $J(\Phi)$ and $J(\Psi)$ are bi-orthogonal if and only if $M_\mu M_\mu{}^\dagger N_\nu N_\nu{}^\dagger = 0$ and $M_\mu{}^\dagger M_\mu N_\nu{}^\dagger N_\nu = 0$ for all μ and ν, if and only if $M_\mu^\dagger N_\nu = 0$ and $M_\mu N_\nu^\dagger = 0$ for all μ and ν. □

By mimicking the Definition 2.1, we introduce the following notion of bi-orthogonal decomposition for CP super-operator:

Definition 4.2. A CP super-operator $\Phi \in \mathsf{T}(\mathcal{H})$ has a *bi-orthogonal decomposition* if $J(\Phi)$ has a bi-orthogonal decomposition: $J(\Phi) = \sum_k D_k$, where $\{D_k\}$ is a family of pairwise bi-orthogonal positive semi-definite operators.

If $J(\Phi)$ can be represented as a sum $\sum_k D_k$ of pairwise bi-orthogonal positive semi-definite operators, decompose each D_k by the spectral decomposition theorem as

$$D_k = \sum_i d_k^{(i)}\,\mathrm{vec}(\widetilde{M}_k^{(i)})\,\mathrm{vec}(\widetilde{M}_k^{(i)})^\dagger = \sum_i \mathrm{vec}(M_k^{(i)})\,\mathrm{vec}(M_k^{(i)})^\dagger,$$

where $M_k^{(i)} \in \mathbf{L}(\mathcal{H})$, $\mathrm{vec}(M_k^{(i)}) = \sqrt{d_k^{(i)}}\,\mathrm{vec}(\widetilde{M}_k^{(i)})$ and $\langle M_k^{(i)}, M_k^{(j)}\rangle = d_k^{(i)}\delta_{ij}$, then $\Phi_k = \sum_i Ad_{M_k^{(i)}}$ as $J(\Phi_k) = D_k$. Since $\mathrm{Tr}_2\, D_k = \sum_i M_k^{(i)} M_k^{(i)\dagger}$ and $\mathrm{Tr}_1\, D_k = \sum_i [M_k^{(i)\dagger} M_k^{(i)}]^\mathsf{T}$, it follows form the bi-orthogonality of $\{D_k\}$ that $M_s^{(i)\dagger} M_t^{(j)} = 0$ and $M_s^{(i)} M_t^{(j)\dagger} = 0$ for any $s \neq t$ and all sub-indices i, j. This implies that $\Phi_m{}^\dagger \circ \Phi_n = 0$ and $\Phi_m \circ \Phi_n{}^\dagger = 0$ if $m \neq n$.

The following proposition can be viewed as a characterization of Φ having a bi-orthogonal decomposition:

Proposition 4.3. *A CP super-operator $\Phi \in \mathsf{T}(\mathcal{H})$ has a bi-orthogonal decomposition if and only if $\Phi = \sum_k \Phi_k$, where $\{\Phi_k\}$ is a collection of CP super-operators in $\mathsf{T}(\mathcal{H})$ and $\Phi_m{}^\dagger \circ \Phi_n = 0$ and $\Phi_m \circ \Phi_n{}^\dagger = 0$ for all $m \neq n$.*

By Proposition 1 in [16], it follows from the above Proposition 4.1 that

1. For $i = 1, 2$, let $\Phi_i, \Psi_i \in \mathsf{T}(\mathcal{H})$ be CP super-operators, Φ_1 and Φ_2 be bi-orthogonal, and Ψ_1 and Ψ_2 be bi-orthogonal. Then for any CP super-operator $\Lambda \in \mathsf{T}(\mathcal{H})$, $\Phi_1 \circ \Lambda \circ \Psi_1$ and $\Phi_2 \circ \Lambda \circ \Psi_2$ are also bi-orthogonal.

2. If $\Phi, \Psi \in \mathsf{T}(\mathcal{H})$ are CP and bi-orthogonal, then for any positive semi-definite operators $X, Y \in \mathbf{L}(\mathcal{H})$, $\Phi(X)$ and $\Psi(Y)$ are orthogonal.

Our mail result of this section is the following:

Theorem 4.4. *Assume that $\Phi, \Lambda, \Psi \in \mathsf{T}(\mathcal{H})$ are CP and bi-stochastic, and the following conditions hold:*

(i) $\mathcal{H} = \bigoplus_{k=1}^K \mathcal{H}_k^L \otimes \mathcal{H}_k^R$, *where* $\dim \mathcal{H}_k^L = d_k^L, \dim \mathcal{H}_k^R = d_k^R$ *and* $\sum_{k=1}^K d_k^L d_k^R = N$;

(ii) $\Phi = \bigoplus_{k=1}^K \Phi_k^L \otimes Ad_{U_k^R}, \Lambda = \bigoplus_{k=1}^K \Lambda_k^L \otimes \Lambda_k^R$, *and* $\Psi = \bigoplus_{k=1}^K Ad_{V_k^L} \otimes \Psi_k^R$,
that is, $\Phi|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = \Phi_k^L \otimes Ad_{U_k^R}, \Psi|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = Ad_{V_k^L} \otimes \Psi_k^R$, *and* $\Lambda|_{\mathbf{L}(\mathcal{H}_k^L \otimes \mathcal{H}_k^R)} = \Lambda_k^L \otimes \Lambda_k^R$, $\Phi_k^L, \Lambda_k^L \in \mathsf{T}(\mathcal{H}_k^L)$ *are CP and bi-stochastic,* $V_k^L \in \mathbf{L}(\mathcal{H}_k^L)$ *are unitary operators,* $U_k^R \in \mathbf{L}(\mathcal{H}_k^R)$ *are unitary operators and* $\Psi_k^R, \Lambda_k^R \in \mathsf{T}(\mathcal{H}_k^R)$ *are CP and bi-stochastic.*

Then we have the following strong dynamical additivity:

$$S(\Phi \circ \Lambda) + S(\Lambda \circ \Psi) = S(\Lambda) + S(\Phi \circ \Lambda \circ \Psi).$$

Proof. Since

$$\Phi \circ \Lambda \circ \Psi = \sum_{k=1}^{K} \Phi_k^L \circ \Lambda_k^L \circ Ad_{V_k^L} \otimes Ad_{U_k^R} \circ \Lambda_k^R \circ \Psi_k^R$$

is a bi-orthogonal decomposition of $\Phi \circ \Lambda \circ \Psi$, it follows that

$$\rho(\Phi \circ \Lambda \circ \Psi) = \sum_{k=1}^{K} \lambda_k \rho(\Phi_k^L \circ \Lambda_k^L \circ Ad_{V_k^L}) \otimes \rho(Ad_{U_k^R} \circ \Lambda_k^R \circ \Psi_k^R),$$

where $\lambda_k = \frac{1}{N} d_k^L d_k^R$ for each k and $\sum_{k=1}^{K} \lambda_k = 1$. Thus,

$$\begin{aligned} \mathsf{S}(\Phi \circ \Lambda \circ \Psi) &= \mathsf{H}(\lambda) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Phi_k^L \circ \Lambda_k^L \circ Ad_{V_k^L}) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(Ad_{U_k^R} \circ \Lambda_k^R \circ \Psi_k^R) \\ &= \mathsf{H}(\lambda) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Phi_k^L \circ \Lambda_k^L) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Lambda_k^R \circ \Psi_k^R). \end{aligned}$$

Similarly,

$$\begin{aligned} \mathsf{S}(\Phi \circ \Lambda) &= \mathsf{H}(\lambda) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Phi_k^L \circ \Lambda_k^L) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Lambda_k^R), \\ \mathsf{S}(\Lambda \circ \Psi) &= \mathsf{H}(\lambda) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Lambda_k^L) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Lambda_k^R \circ \Psi_k^R), \\ \mathsf{S}(\Lambda) &= \mathsf{H}(\lambda) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Lambda_k^L) + \sum_{k=1}^{K} \lambda_k \mathsf{S}(\Lambda_k^R), \end{aligned}$$

where $\mathsf{H}(\lambda) = -\sum_{k=1}^{K} \lambda_k \log_2 \lambda_k$ is the *Shannon entropy* of $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_K)$. It follows from these equalities that $\mathsf{S}(\Phi \circ \Lambda) + \mathsf{S}(\Lambda \circ \Psi) = \mathsf{S}(\Lambda) + \mathsf{S}(\Phi \circ \Lambda \circ \Psi)$. □

5 Concluding Remarks

In a closed quantum system, clearly $\Phi = Ad_U, \Lambda = Ad_V, \Psi = Ad_W$ can saturate the Strong Dynamical Subadditivity(SDS), where U, V, W are unitary operators. Thus this is trivial case. More generally, in an open quantum system, there is a complete different scenario. In order to saturate the SDS, local operation—today commonly called *no-signaling* [6]—could be considered in this case. It can be seen from the Theorem 4.4 that when Φ, Λ, Ψ are all local operations, then SDS is saturated by no-signaling operations. Hence the underlying Hilbert space and corresponding quantum operations can be viewed as a bipartite space and bipartite operations. Intuitively, a quantum operation is no-signaling if it cannot be used by spatially separated parties to violate relativistic causality, i.e., no-signaling quantum operation jointly implemented by several parties that cannot use it to communicate with each other. Therefore, the entropy of the composite quantum operations is just changed locally. The sufficient condition in Theorem 4.4 is supported by the impossibility of communicating by local operations. It is conjectured that SDS cannot be saturated by non-local operations. So looking for a necessary condition to Theorem 4.4 may be restricted within the set of no-signalling operations.

A possible application can be expected by the following consideration. Firstly, we recall some concepts for quantum states. The so-called *squashed entanglement* [5] are proposed recently by Christandl *et. al.* and some attractive properties of it are established. Among all known entanglement measures, squashed entanglement is the entanglement measure which satisfies most properties that have been proposed as useful for an entanglement

measure [2]. The squashed entanglement is related to the strong subadditivity of entropy for quantum states. It is described by the quantity

$$E_{sq}(\rho^{AB}) = \inf_{E}\{\frac{1}{2}I(A;B|E) : \rho^{ABE} \text{ extension of } \rho^{AB}\},$$

where

$$I(A;B|E) = \mathsf{S}(\rho^{AE}) + \mathsf{S}(\rho^{BE}) - \mathsf{S}(\rho^{ABE}) - \mathsf{S}(\rho^{E})$$

is the quantum conditional mutual information of ρ^{ABE}, which measures the correlations of two quantum systems relative to a third one. One important property of E_{sq} is that it is faithful [2]: $E_{sq}(\rho^{AB}) = 0$ if and only if ρ^{AB} is separable state. Based on this result, an approximate version of the fact are obtained that states ρ^{ABE} with zero conditional mutual information $I(A;B|E)$ are such that ρ^{AB} is separable, that is, if a tripartite state has small conditional mutual information, its AB reduction is close to a separable state. This problem is left open at the end of [8]. The conditional mutual information $I(A;B|E)$ is also used to demarcates the edges of quantum correlations by Datta [7].

The above developments motivate naturally us to consider analogous problems for quantum operations. For instance, for the given quantum operations $\Phi,\Lambda,\Psi \in \mathsf{T}(\mathcal{H})$, when they are all bi-stochastic, the quantity

$$I(\Phi;\Psi|\Lambda) = \mathsf{S}(\Phi\circ\Lambda) + \mathsf{S}(\Lambda\circ\Psi) - \mathsf{S}(\Phi\circ\Lambda\circ\Psi) - \mathsf{S}(\Lambda)$$

can be defined similarly, but unfortunately which is asymmetric with respect to the pair (Φ,Ψ), compared with the quantum state situation. This is clear since different composite ordering of quantum operations lead to different magnitudes. Apparently, there are two extreme quantity

$$\sup_{\Lambda}\{I(\Phi;\Psi|\Lambda) : \Lambda \in \mathsf{T}(\mathcal{H}) \text{ being CP and bi-stochastic}\},$$

and

$$\inf_{\Lambda}\{I(\Phi;\Psi|\Lambda) : \Lambda \in \mathsf{T}(\mathcal{H}) \text{ being CP and bi-stochastic}\}$$

can be considered. They may signify the maximal/minimum capacity of decoherence induced by the composition of quantum operations Φ and Ψ. We leave it for the future research.

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